MATH 105A and 110A Review: Eigenvalues and diagonalization

1. Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

Solution: To find the eigenvalues, we find λ such that

$$0 = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2.$$

Thus, $\lambda = 2$ is an eigenvalue with multiplicity 2. This means that A is diagonalizable if and only if there are 2 linearly independent eigenvectors corresponding to $\lambda = 2$. To find eigenvectors, we reduce the following to reduced echelon form:

$$A - 2 \cdot I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} 1/3 \cdot R_1 \to R_1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence, all eigenvectors x are of the form:

$$x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence, we cannot find two linearly independent eigenvectors. Thus, A is not diagnonalizable.

2. Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: To find the eigenvalues, we find λ such that

$$0 = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda).$$

Thus, $\lambda = 1$ is an eigenvalue with multiplicity 2 and $\lambda = 3$ is an eigenvalue with multiplicity 1. This means that A is diagonalizable if and only if there are 2 linearly independent eigenvectors corresponding to $\lambda = 1$ and there are 1 eigenvectors corresponding to $\lambda = 1$ (note that there is always at least 1 eigenvector). To find eigenvectors corresponding to $\lambda = 1$, we need to find x such that

$$0 = (A - 1 \cdot I)x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix})x.$$

Hence,

$$x = \begin{pmatrix} x_1 \\ (-3/2)x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -3/2 \\ 1 \end{pmatrix}.$$

Thus, the two linearly independent eigenvectors corresponding to $\lambda = 1$ are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -3/2 \\ 1 \end{pmatrix}.$$

To find eigenvectors corresponding to $\lambda = 3$, we need to find x such that

$$0 = (A - 3 \cdot I)x = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} x.$$

$(A - 3 \cdot I)$ row reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda = 3$. Thus, the diagonalization is

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/2 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3/2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3/2 \end{bmatrix}. \end{aligned}$$